Theory of stellar distortion by fast rotators

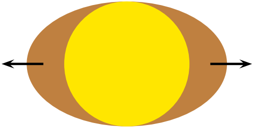
Francisco Espinosa Lara
francisco.espinosa@irap.omp.eu

IRAP, Toulouse

September 17, 2013

What is a fast rotator?

- A fast rotator is a star with a rotation velocity large enough to affect its structure.
- The centrifugal force is comparable to the gravity.
- Fast rotation affects the shape of the star and its brightness.



Why do stars rotate?

- Stars are formed by the collapse of a molecular cloud.
  - Specific angular momentum of the molecular cloud

$$j_{\rm cloud} = \delta v_{\rm cloud} R_{\rm cloud} \sim 0.1 {\rm km/s} \cdot 10^{17} {\rm cm} = 10^{21} {\rm cm}^2/{\rm s}$$

  - Specific angular momentum of the star

$$j_\star = \Omega_\star R_\star^2$$

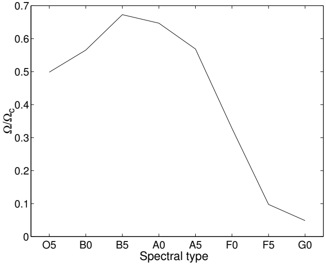
  - If angular momentum is conserved during the collapse $j_{\rm cloud} = j_\star$

$$\Omega_\star \sim 10^3 \Omega_{\rm bk} \quad !!!$$

Rotation in stars

- Braking mechanisms
  - **Magnetic braking.** Magnetic interaction with surrounding matter (disk, wind).
  - **Structural changes.** Red giants.
  - **Tidal interactions.**

- Main sequence stars with spectral types from O to early F use to be fast rotators.



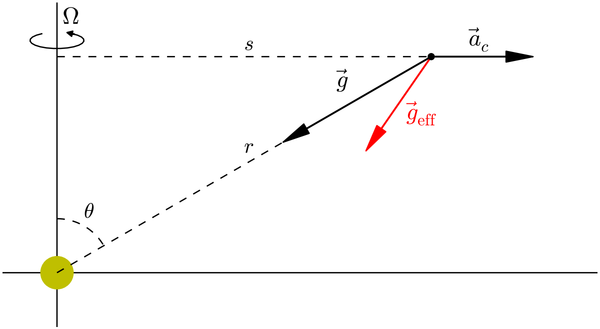
Modelling rotating stars

- The shape of the star is distorted by the centrifugal force.
- Mechanical and thermal equilibrium in (at least) 2 dimensions (r,θ).
- Microscopic processes (nuclear energy generation, energy transport, equation of state)
- Dynamics:
  - Differential rotation
  - Meridional currents
- Turbulent transport:
  - Chemical elements
  - Energy
  - Angular momentum

The Roche model

- Approximate the gravitational field inside a rotating star by the field created by a point mass located in the stellar centre in a rotating frame.
- It is equivalent to considering that all the mass of the star is concentrated at the centre.
- Gives a good approximation of the shape of the star and the surface gravity.

The Roche model



$$\underbrace{\vec{g}_{\rm eff}}_{\text{Effective gravity}} = \underbrace{-\frac{GM}{r^2}\hat{r}}_{\text{Gravity } (\vec{g})} + \underbrace{\Omega^2 r \sin\theta \hat{s}}_{\text{Centrifugal acc. } (\vec{a}_c)}$$

The Roche model

$\vec{g}_{\text{eff}} = -\frac{GM}{r^2}\hat{r} + \Omega^2 r \sin\theta \hat{s}$

- The resulting force is conservative and can be written as the gradient of a potential

$$\vec{g_{\text{eff}}} = -\nabla\phi_{\text{eff}}$$
$$\phi_{\text{eff}} = -\frac{GM}{r} - \frac{1}{2}\Omega^2 r^2 \sin^2\theta$$

- The surface of the star is an equipotential surface ($\phi_{\text{eff}} =$const.), then

$$\frac{GM}{r} + \frac{1}{2}\Omega^2 r^2 \sin^2\theta = \text{const.} \quad \text{at the surface}$$

Equation of the surface

$\frac{GM}{r} + \frac{1}{2}\Omega^2 r^2 \sin^2\theta = \text{const.}$ at the surface

- Normalization: $\tilde{r} = \dfrac{r}{R_\mathrm{e}}$

$$\frac{1}{\tilde{r}} + \frac{1}{2}\Omega^2 \frac{R_\mathrm{e}^3}{GM}\tilde{r}^2 \sin^2\theta = \text{const.}$$

- Let's define $\Omega_k = \sqrt{\frac{GM}{R_\mathrm{e}^3}}$ and $\omega = \frac{\Omega}{\Omega_k}$

$$\frac{1}{\tilde{r}} + \frac{1}{2}\omega^2 \tilde{r}^2 \sin^2\theta = \text{const.}$$

- This expression is constant across the surface. In particular, at the equator ($\tilde{r} = 1,\ \theta = \pi/2$): const.$= 1 + \frac{1}{2}\omega^2$

Equation of the surface

$$\frac{1}{\tilde{r}} + \frac{1}{2}\omega^2 \tilde{r}^2 \sin^2\theta = 1 + \frac{1}{2}\omega^2$$

Meaning of Ω_k

$\Omega_k = \sqrt{\frac{GM}{R_\mathrm{e}^3}}$

- Remember the expression of the effective gravity

$$\vec{g}_\mathrm{eff} = -\frac{GM}{r^2}\hat{r} + \Omega^2 r \sin\theta \hat{s}$$

- At the equator $r = R_\mathrm{e}$, $\theta = \pi/2$ and $\hat{r} = \hat{s}$

$$\vec{g}^\mathrm{e}_\mathrm{eff} = \left(-\frac{GM}{R_\mathrm{e}^2} + \Omega^2 R_\mathrm{e}\right)\hat{s}$$

- If $\Omega = \Omega_k$

$$\vec{g}^\mathrm{e}_\mathrm{eff} = \left(-\frac{GM}{R_\mathrm{e}^2} + \frac{GM}{R_\mathrm{e}^3} R_\mathrm{e}\right)\hat{s} = \vec{0}$$

- Ω_k is called critical (keplerian) velocity. For $\Omega > \Omega_k$, a parcel of gas at the equator will be no longer bounded to the star.

Solving the equation of the surface

$\frac{1}{\tilde{r}} + \frac{1}{2}\omega^2\tilde{r}^2\sin^2\theta = 1 + \frac{1}{2}\omega^2$

- For a given value of $\omega \in [0,1]$ we want to solve $r = r(\theta)$.
- **Newton's method.**
  - Rewrite the equation as

$$h_{\omega,\theta}(r) = w^2 r^3 \sin^2\theta - (2+\omega^2)r + 2 = 0$$

  - Solve by iteration

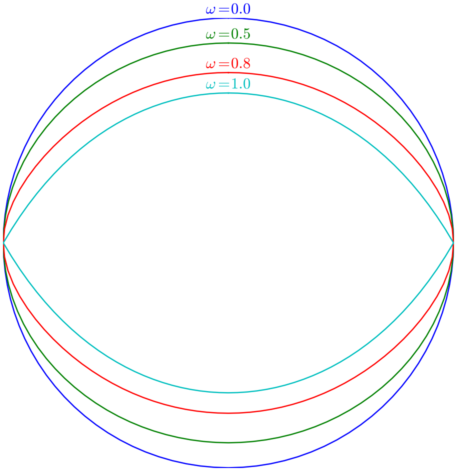
$$r_{k+1} \longleftarrow r_k - \frac{h_{\omega,\theta}(r_k)}{h'_{\omega,\theta}(r_k)}$$

  for each θ with $r_0 = 1$ until convergence

$$r_{k+1} \longleftarrow r_k - \frac{w^2 r_k^3 \sin^2\theta - (2+\omega^2)r_k + 2}{3w^2 r_k^2 \sin^2\theta - (2+\omega^2)}$$

Solving the equation of the surface

$$\frac{1}{\tilde{r}} + \frac{1}{2}\omega^2\tilde{r}^2\sin^2\theta = 1 + \frac{1}{2}\omega^2$$



Flattening

$\frac{1}{\tilde{r}} + \frac{1}{2}\omega^2\tilde{r}^2\sin^2\theta = 1 + \frac{1}{2}\omega^2$

- The flattening is defined as

$$\epsilon = \frac{R_\mathrm{e} - R_\mathrm{p}}{R_\mathrm{e}} = 1 - \tilde{r}_p$$

- Using the equation of the surface for the pole $(\theta = 0)$

$$\tilde{r}_p = \frac{2}{\omega^2 + 2}$$

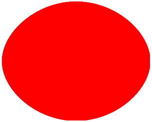
$$\epsilon = \frac{\omega^2}{\omega^2 + 2}$$

- For Roche models, the maximum value for the flattening is $(\omega = 1)$

$$\epsilon_\mathrm{max} = \frac{1}{3} \qquad (R_\mathrm{e} = 1.5R_\mathrm{p})$$

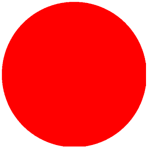
Inclination angle

- The inclination angle i is the angle formed by the rotation axis and the line of sight
- $i = 90^\circ$ (Equator-on)



$v \sin i = \Omega R_\mathrm{e}$

- $i = 0^\circ$ (Pole-on)



$v \sin i = 0$

Another critical velocity

$\Omega_k = \sqrt{\frac{GM}{R_\mathrm{e}^3}}$

- What is the maximum angular velocity for a given R_p?

$$\Omega_c = \sqrt{\frac{GM}{R_\mathrm{e,max}^3}}, \qquad R_\mathrm{e,max} = 1.5 R_\mathrm{p} \quad \longrightarrow \quad \boxed{\Omega_c = \sqrt{\frac{8}{27}\frac{GM}{R_\mathrm{p}^3}}}$$

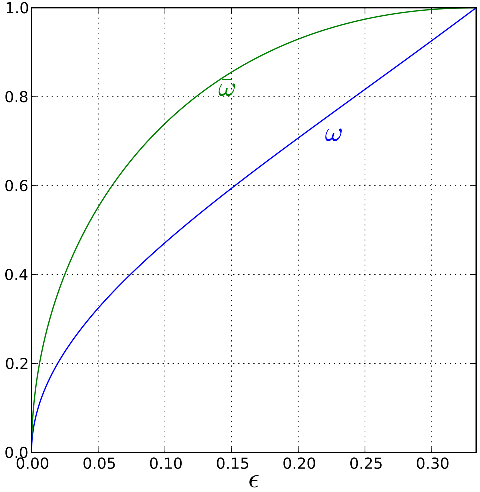
- For a given model $(M, R_\mathrm{p}, R_\mathrm{e})$ $\Omega_k \neq \Omega_c$

Ω_c (Critical velocity):	Ω of a Roche model with the same M and R_p rotating with critical velocity
Ω_k (Keplerian velocity):	Needed Ω to "escape" from the equator of the star (at current R_e)

- Let's define $\bar{\omega} = \frac{\Omega}{\Omega_c}$

$$\bar{\omega}^2 = \frac{\Omega_k^2}{\Omega_c^2}\omega^2 = \left(\frac{3}{2+\omega^2}\right)^3 \omega^2$$

ω & $\bar{\omega}$ vs. flattening



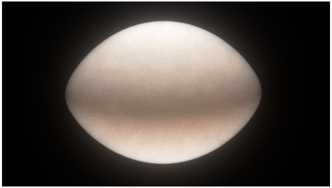
$$\omega = \frac{\Omega}{\Omega_k} \qquad \Omega_k = \sqrt{\frac{GM}{R_\mathrm{e}^3}}$$

$$\bar{\omega} = \frac{\Omega}{\Omega_c} \qquad \Omega_c = \sqrt{\frac{8}{27}\frac{GM}{R_\mathrm{p}^3}}$$

$$\bar{\omega}^2 = \left(\frac{3}{2+\omega^2}\right)^3 \omega^2$$

Gravity darkening

- The Roche model gives the shape of the star but, what about its brightness?
    - **Gravity darkening**. Rotating stars are brighter at their poles.



- Estimate the flux of energy, or equivalently $T_{\rm eff}$, at the surface of a rotating star.

$$\hat{n} \cdot \vec{F} = \sigma T_{\rm eff}^4$$

Von Zeipel's model (1924)

- Uniform rotation.
- Energy is transported by radiation only.
- **Barotropic stratification**.

  ρ, T, depend only on the potential ϕ.

Using this assumptions:

$$\vec{F} = -\chi \nabla T = -\chi(\phi) T'(\phi) \nabla \phi_{\text{eff}} = \chi(\phi) T'(\phi) \vec{g}_{\text{eff}}$$

$$F \propto g_{\text{eff}}$$

$$\boxed{T_{\text{eff}} \propto g_{\text{eff}}^{0.25}}$$ Von Zeipel's law

Von Zeipel's model (1924)

But. . .

- The hypothesis of barotropicity is not compatible with thermal equilibrium.
- Real rotating stars are not barotropic.
  - Differential rotation
  - Meridional currents
- Observations do not support von Zeipel's law

Modifications of von Zeipel's law

Lucy (1967). Gravity darkening in stars with convective envelopes.

- Searches for a relation $T_{\rm eff}(g_{\rm eff})$ at constant entropy calibrated using non-rotating models with convection.
- Extrapolates this relation along the surface of rotating stars.

$$T_{\rm eff} \propto g_{\rm eff}^{0.08}$$

However...

- There is no universal relation $T_{\rm eff}(g_{\rm eff})$. $T_{\rm eff}$ and $g_{\rm eff}$ are not local state variables, they depend on the conditions in the interior of the star.
- A rotating star cannot be modelled as a sequence of non-rotating 1D models for each latitude. It would not be in hydrostatic equilibrium.

Generalized law

- Generalized gravity darkening law:

$$T_{\mathrm{eff}} \propto g_{\mathrm{eff}}^{\beta}$$

  - Von Zeipel: $\beta = 0.25$ (radiative)
  - Lucy: $\beta = 0.08$ (convective)
- Value of β:
  - Estimated from the local conditions in the external layers (radiation+convection).
    - The structure of the external layers depends on the flux that is coming from the interior and not the opposite.
  - Treated as a free parameter.
    - Introduces an unnecessary additional unknown with no physical meaning.

New hypotheses

- Although the structure is not barotropic, there is a well-defined vertical direction given by $\vec{g}_{\text{eff}}$.

$$\text{Horizontal length scale} \gtrsim 10^3 \text{ Vertical length scale}$$

- Consider that the energy flux $(\vec{F})$ has the direction of $\vec{g}_{\text{eff}}$.

$$\vec{F} = -f(r,\theta)\vec{g}_{\text{eff}}$$

- Reduces the problem to one scalar quantity f.

New hypotheses

$\vec{F} = -f(r,\theta)\vec{g}_{\rm eff}$

- Consider that all the energy is produced at the centre ($r = 0$) of the star.
- Everywhere in the star (except $r = 0$):

$$\nabla \cdot \vec{F} = 0$$

  - General conservation equation
  - Independent of the mechanism of energy transport (radiation, convection,. . .)
- Combining with the previous hypothesis

$$f\nabla \cdot \vec{g}_{\rm eff} + \vec{g}_{\rm eff} \cdot \nabla f = 0$$

- f depends only on $\vec{g}_{\rm eff}$,

Simplifying things

$f \nabla \cdot \vec{g}_{\mathrm{eff}} + \vec{g}_{\mathrm{eff}} \cdot \nabla f = 0$

Further simplifications:

- Uniform rotation (Ω =const.).
- Roche model

$$\vec{g}_{\mathrm{eff}} = -\frac{GM}{r^2}\hat{r} + \Omega^2 r \sin\theta \hat{s}$$

$$\nabla \cdot \vec{g}_{\mathrm{eff}} = 2\Omega^2$$

- Using the same normalization as before ($\tilde{r} = \frac{r}{R_{\mathrm{e}}}$, $\omega = \frac{\Omega}{\Omega_k}$) and $\tilde{f} = \frac{4\pi GM}{L} f$

$$\left(\frac{1}{w^2\tilde{r}^2} - \sin^2\theta\right)\frac{\partial \tilde{f}}{\partial \tilde{r}} - \sin\theta\cos\theta\frac{\partial \tilde{f}}{\partial \theta} - 2\tilde{f} = 0$$

- Condition on the total luminosity

$$\lim_{r\to 0} \tilde{f} = 1$$

Solving for $\tilde{f}$

$$\left(\frac{1}{w^2\tilde{r}^2} - \sin^2\theta\right)\frac{\partial\tilde{f}}{\partial\tilde{r}} - \sin\theta\cos\theta\frac{\partial\tilde{f}}{\partial\theta} - 2\tilde{f} = 0 \qquad \lim_{r\to 0}\tilde{f} = 1$$

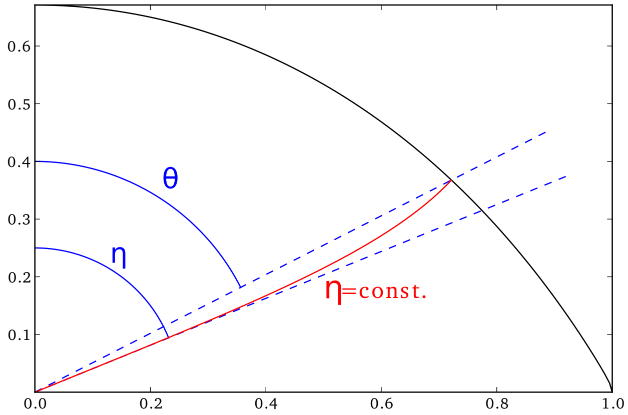
- This equation can be solved (see Espinosa Lara & Rieutord 2011 for details)

$$\tilde{f} = \frac{\tan^2\eta}{\tan^2\theta}$$

where η is defined from the expression

$$\cos\eta + \ln\tan\frac{\eta}{2} = \frac{1}{3}\omega^2\tilde{r}^3\cos^3\theta + \cos\theta + \ln\tan\frac{\theta}{2}$$

- $\tilde{f}$ depends only on ω.


0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.2
0.4
0.6
0.8
1.0
θ
η
η=const.

Calculating $\tilde{f}$

- Get $\tilde{r}$ from the Roche model
- Use Newton's method to calculate η for each θ

$$\cos\eta + \ln\tan\frac{\eta}{2} = \frac{1}{3}\omega^2\tilde{r}^3\cos^3\theta + \cos\theta + \ln\tan\frac{\theta}{2}$$

$$\eta_{k+1} \longleftarrow \eta_k - \frac{\cos\eta + \ln\tan\frac{\eta}{2} - \frac{1}{3}\omega^2\tilde{r}^3\cos^3\theta - \cos\theta - \ln\tan\frac{\theta}{2}}{-\sin\eta + \frac{1}{\sin\eta}}$$

  with $\eta_0 = \theta$.
- Get $\tilde{f}$

$$\tilde{f} = \frac{\tan^2\eta}{\tan^2\theta}$$

- At the pole and the equator

$$\tilde{f}_\mathrm{p} = \mathrm{e}^{\frac{2}{3}\omega^2\tilde{r}_p^3} \qquad \tilde{f}_\mathrm{e} = (1-\omega^2)^{-2/3}$$

Effective temperature

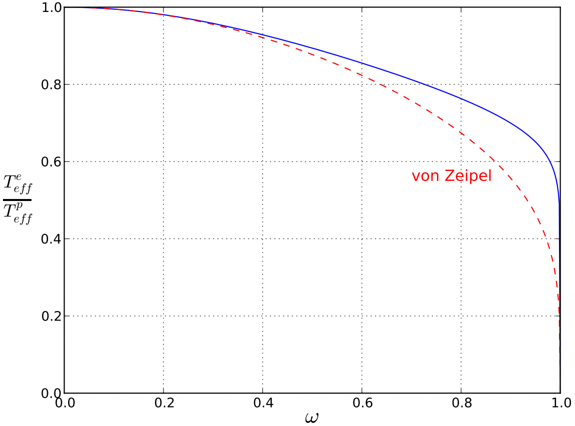
$\tilde{f} = \frac{\tan^2 \eta}{\tan^2 \theta}$

- Effective temperature

$$T_{\text{eff}} = \left(\frac{F}{\sigma}\right)^{0.25} = \left(\frac{L}{4\pi\sigma GM}\right)^{0.25} \left(\tilde{f}(\theta) g_{\text{eff}}\right)^{0.25}$$

- For slow rotation $\eta \sim \theta$, $\tilde{f} \sim 1$, we recover von Zeipel's law.

Effective temperature



$$\frac{T_{\rm eff}^{\rm e}}{T_{\rm eff}^{\rm p}} = \left(\frac{\tilde{f}_{\rm e}}{\tilde{f}_{\rm p}}\frac{g_{\rm eff}^{\rm e}}{g_{\rm eff}^{\rm p}}\right)^{0.25} = \sqrt{\frac{2}{2+\omega^2}}\left(1-\omega^2\right)^{1/12}\exp\left(-\frac{4}{3}\frac{\omega^2}{(2+\omega^2)^3}\right)$$

Summary of the model

Dimensionless model

- Start with a value of $\epsilon = 1 - \frac{R_\mathrm{p}}{R_\mathrm{e}}$
- Calculate $\omega = \sqrt{\dfrac{2\epsilon}{1-\epsilon}}$
- Get the shape of the surface $\tilde{r}(\theta)$:

$$\frac{1}{\tilde{r}} + \frac{1}{2}\omega^2\tilde{r}^2\sin^2\theta = 1 + \frac{1}{2}\omega^2$$

- Calculate $\tilde{g}_{\mathrm{eff}}(\theta) = \sqrt{\dfrac{1}{\tilde{r}^4} + \omega^4\tilde{r}^2\sin^2\theta - \dfrac{2\omega^2\sin^2\theta}{\tilde{r}}}$
- Get $\eta(\theta)$:

$$\cos\eta + \ln\tan\frac{\eta}{2} = \frac{1}{3}\omega^2\tilde{r}^3\cos^3\theta + \cos\theta + \ln\tan\frac{\theta}{2}$$

- Calculate $\tilde{T}_{\mathrm{eff}}(\theta) = \sqrt{\dfrac{\tan\eta}{\tan\theta}}\;\tilde{g}_{\mathrm{eff}}^{0.25}$

Summary of the model

Recovering dimensions

- We know M, R_{e} and L.
- Angular velocity:

$$\Omega = \omega \Omega_k = \omega \sqrt{\frac{GM}{R_{\mathrm{e}}^3}}$$

- Shape of the surface:

$$r(\theta) = \tilde{r}(\theta) R_{\mathrm{e}}$$

- Surface gravity:

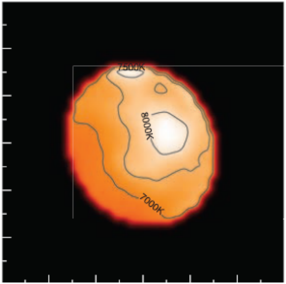
$$g_{\mathrm{eff}} = \tilde{g}_{\mathrm{eff}} \frac{GM}{R_{\mathrm{e}}^2}$$

- Effective temperature:

$$T_{\mathrm{eff}} = \tilde{T}_{\mathrm{eff}} \left(\frac{L}{4\pi\sigma R_{\mathrm{e}}^2} \right)^{0.25}$$

Modelling Altair

- Altair (α Aql) is the brightest star in the constellation Aquila ($m_v = 0.77$).
- A-type star with $\sim 1.8 M_{\odot}$.
- Variable star of type δ-Scuti.
- Observed by Monnier et al 2007 using the CHARA interferometer.



Modelling Altair

Monnier et al 2007

$R_\mathrm{p} = 1.634 \pm 0.011\, R_\odot \qquad T_\mathrm{eff}^\mathrm{p} = 8450 \pm 140$ K

$R_\mathrm{e} = 2.029 \pm 0.007\, R_\odot \qquad T_\mathrm{eff}^\mathrm{e} = 6860 \pm 150$ K

- Flattening: $\epsilon = 1 - \dfrac{R_\mathrm{p}}{R_\mathrm{e}} = 0.1947$
- Angular velocity: $\omega = \sqrt{\frac{2\epsilon}{1-\epsilon}} = 0.695$

$$\Omega = 0.695\,\Omega_k = 0.923\,\Omega_c$$

- Effective temperature:

$$\frac{T_\mathrm{eff}^\mathrm{e}}{T_\mathrm{eff}^\mathrm{p}} = 0.814 \qquad \text{(Monnier et al: 0.812)}$$

Von Zeipel: 0.761

Modelling Altair

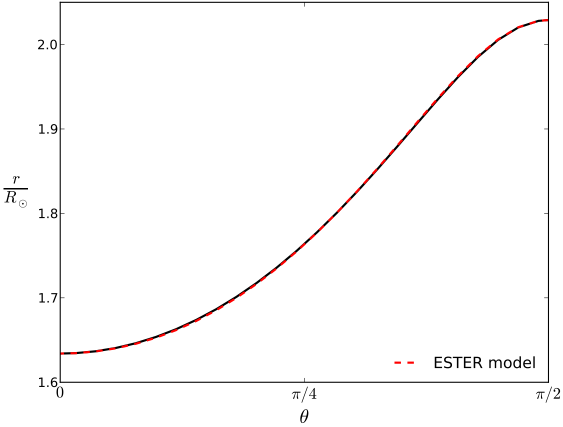
Comparison with ESTER model

- Comparison model calculated with the 2D stellar evolution code ESTER (`https://code.google.com/p/ester-project/`)
- Model parameters

	ESTER model	Monnier et al 2007
Mass	$1.76\,M_\odot$	
Luminosity	$10.32\,L_\odot$	
R_p (input)	$1.634\,R_\odot$	$1.634 \pm 0.011\,R_\odot$
R_e (input)	$2.029\,R_\odot$	$2.029 \pm 0.007\,R_\odot$
$T_\mathrm{eff}^\mathrm{p}$ (input)	8450 K	8450 ± 140 K
$T_\mathrm{eff}^\mathrm{e}$	6868 K	6860 ± 150 K
v_e	281 km/s	285 km/s

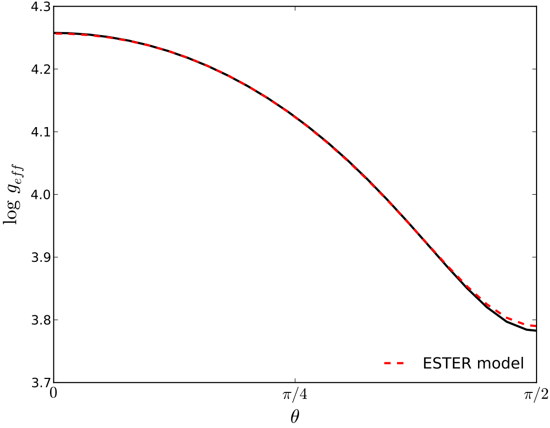
Modelling Altair
Comparison with ESTER model

- Radius



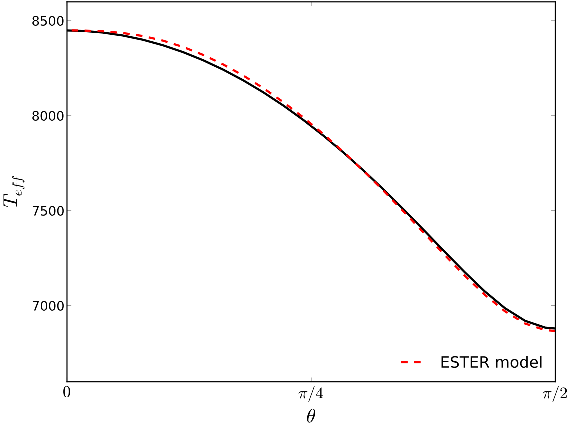
Modelling Altair
Comparison with ESTER model

- Surface gravity

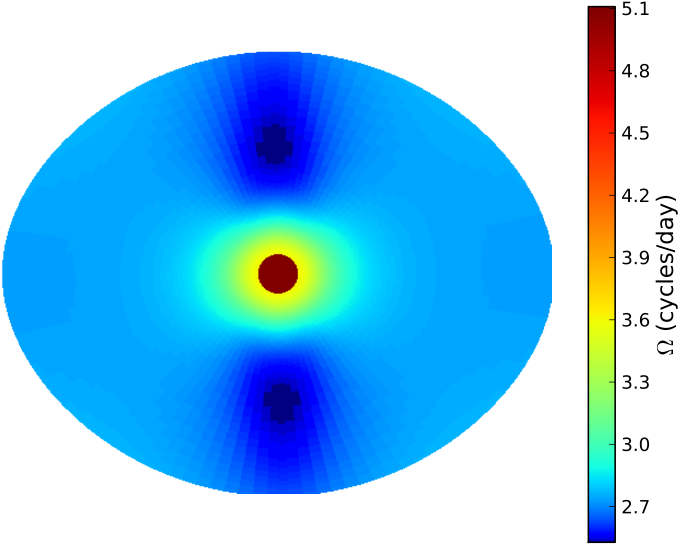


Modelling Altair
Comparison with ESTER model

- Effective temperature



Differential rotation from ESTER model



Conclusions

- A simple model Roche + gravity darkening gives a good approximation of the shape of the star, as well as the profiles of $g_{\rm eff}$ and $T_{\rm eff}$ at the surface of a rotating star.
- The normalized profiles depend only on a single parameter (ω or ϵ).
- Deviations from the model come from:
  - Mass distribution.
  - Energy generation.
  - Differential rotation.
- Limitations of the model:
  - Gives no information on the internal structure of the star (ρ, T, p).
  - Cannot reproduce the differential rotation profile
  - ...