Classic disc physics

II - simple analytical models

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Lectures outline

- Lecture I - Basic equations
- Lecture II - Steady state solutions - Time dependent models - Outbursts
- Lecture III - Sources of angular momentum transport

Steady state solutions

- Continuity:

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{R}\frac{\partial}{\partial R}(R\Sigma u) = 0$$

- In a steady state:

$$\dot{M} = -2\pi R\Sigma u$$

Steady state solutions

$$\dot{M} = -2\pi R \Sigma u$$

- Angular momentum conservation:

$$\frac{\partial}{\partial t}(\Sigma \Omega R^2) + \frac{1}{R}\frac{\partial}{\partial R}(\Sigma \Omega R^3 u) = -\frac{1}{R}\frac{\partial}{\partial R}(\nu \Sigma R^3 |\Omega'|)$$

- In a steady state:

$$\dot{M} j(R_{\rm in}) = \dot{M} \Omega R^2 - 2\pi \nu \Sigma R^3 \Omega'$$

Steady state solutions

$$\dot{M} = -2\pi R \Sigma u$$

- Angular momentum conservation:

$$\frac{\partial}{\partial t}(\Sigma \Omega R^2) + \frac{1}{R}\frac{\partial}{\partial R}(\Sigma \Omega R^3 u) = -\frac{1}{R}\frac{\partial}{\partial R}(\nu \Sigma R^3 |\Omega'|)$$

- In a steady state:

$$\dot{M} j(R_{\rm in}) = \dot{M} \Omega R^2 + 3\pi \nu \Sigma R^2 \Omega$$

Steady state solutions

$$\dot{M} = -2\pi R \Sigma u$$

- Angular momentum conservation:

$$\frac{\partial}{\partial t}(\Sigma\Omega R^2) + \frac{1}{R}\frac{\partial}{\partial R}(\Sigma\Omega R^3 u) = -\frac{1}{R}\frac{\partial}{\partial R}(\nu\Sigma R^3|\Omega'|)$$

- In a steady state:

$$3\pi\nu\Sigma = \dot{M}\left[1 - \sqrt{\frac{R_{\rm in}}{R}}\right]$$

Some comments

- Far from inner boundary

$$\dot{M} = 3\pi\nu\Sigma$$

- If viscosity increases, say, linearly with R, in a steady state Σ decreases linearly

Velocity ordering

- Radial velocity: $u \sim -\nu/R$
- Popular prescription (Shakura & Sunyaev 1973):

$$\nu = \alpha c_{\mathrm{s}} H = \alpha \left(\frac{H}{R}\right)^2 \Omega R^2$$

- Ordering

$$u = \alpha (H/R)^2 v_\phi \ll c_{\mathrm{s}} = (H/R) v_\phi \ll v_\phi$$

Energetics

- During accretion, gravitational binding energy is released
- Dissipated by viscosity
- Rate of viscous dissipation

$$D(R) = \nu\Sigma(R\Omega')^2 = \frac{3}{4\pi}\frac{GM\dot{M}}{R^3}\left[1-\sqrt{\frac{R_{\mathrm{in}}}{R}}\right]$$

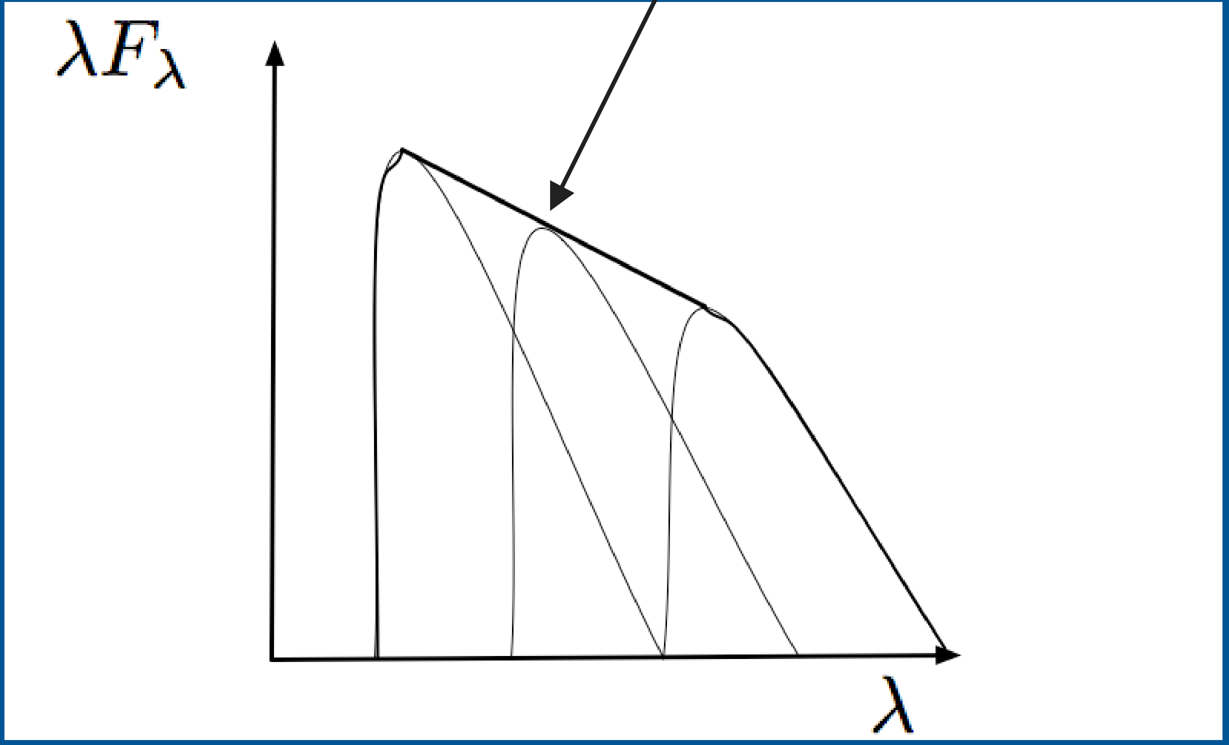
- If this energy is released as a blackbody by the disc surface

$$\sigma T_{\mathrm{s}}^4 = \frac{3}{8\pi}\frac{GM\dot{M}}{R^3}\left[1-\sqrt{\frac{R_{\mathrm{in}}}{R}}\right]$$

Note factor 1/2:
disc has two sides

Simple blackbody disc spectrum

- Easy to show:
  - If temperature profile $T(R) \sim T^{-q}$, then
  - SED $\lambda F(\lambda) \sim \lambda^{-(4-2/q)} \sim \lambda^{-(4/3)}$



See Sebastian Wolf lectures

More on energetics

- Total energy dissipated by the disc

$$\int_{R_{\rm in}}^{\infty} 2\pi R D(R) \mathrm{d}R = \frac{1}{2}\frac{GM\dot{M}}{R_{\rm in}}$$

  – Only half of the potential energy, where did the other half go?

- Energy dissipated at a given radius (width dR)

$$2\pi R D(R) \mathrm{d}R = \frac{3}{2}\frac{GM\dot{M}}{R^2}\left[1 - \sqrt{\frac{R_{\rm in}}{R}}\right] \mathrm{d}R$$

  – Another puzzle: only $GMM\mathrm{d}R/2R^2$ comes from gravitational binding energy: where does the rest come from?

Time dependent models

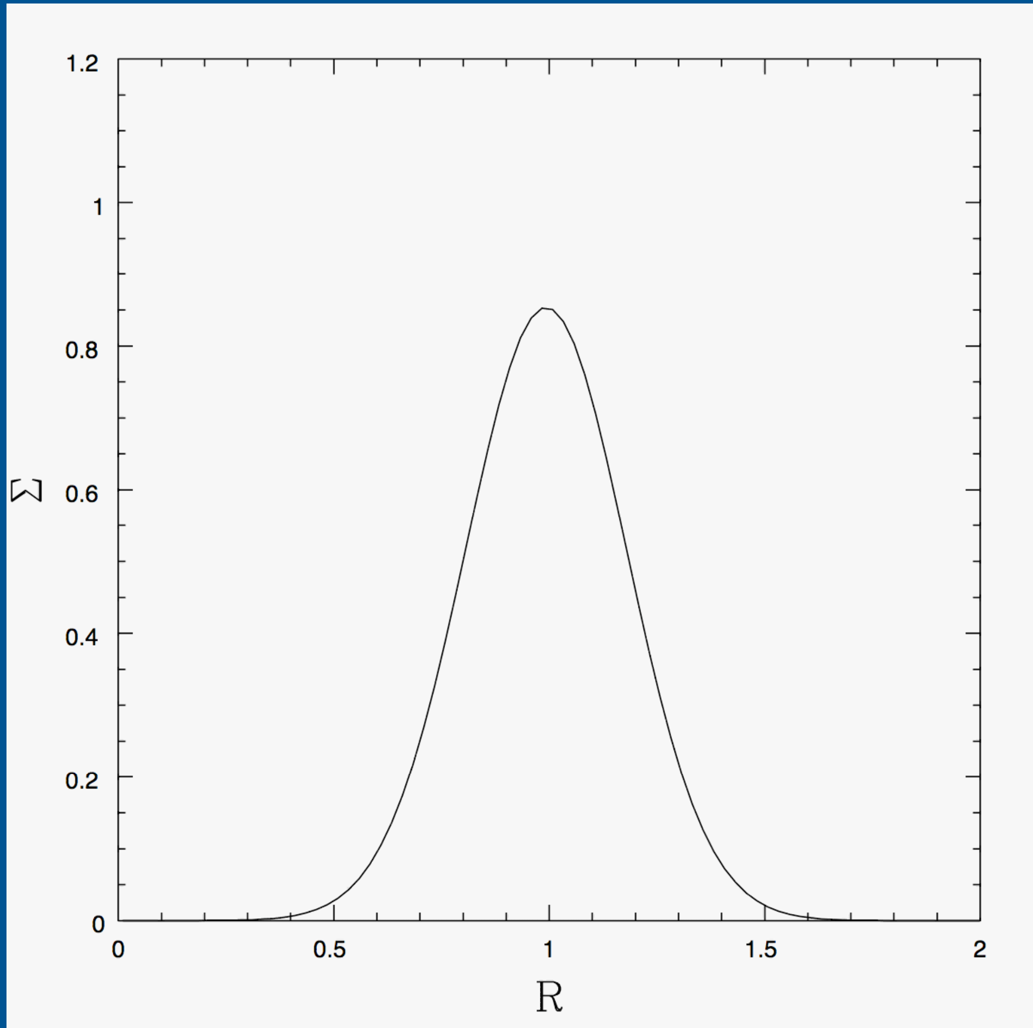
I - spreading ring

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left[R^{1/2}\frac{\partial}{\partial R}(\Sigma \nu R^{1/2})\right]$$

- It is possible to solve analitically in a few simple cases
  - Initially infinitesimally narrow ring (delta-function) -- Pringle (1981)
  - Constant viscosity

Time dependent models

I - spreading ring



Time dependent models

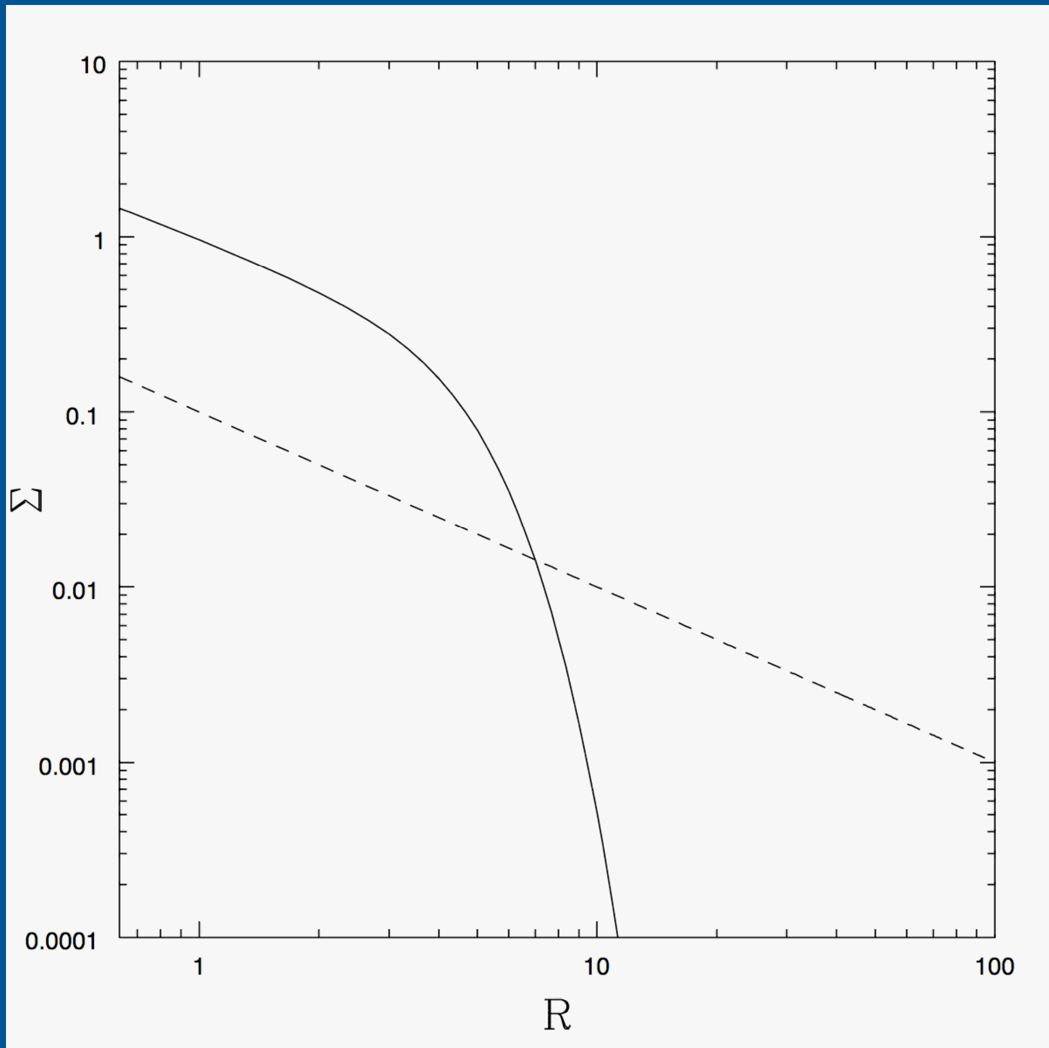
- Viscous timescale $t_{\mathrm{visc}} \sim R^2/\nu \quad (\sim R/u)$
- Note: most often this timescale is a strongly increasing function of radius
- Evolution at large distances much slower
- Note: inner parts move in, outer parts move out!
- It is not just accretion, it is spreading!
- Transition from inward to outward motion at $R \sim R_0 \left(\frac{t}{t_{\mathrm{visc},0}}\right)^a$
- At large times, most of the disc moves inwards

- An infinitesimally small amount of matter reaches infinitely large distances to carry away the angular momentum

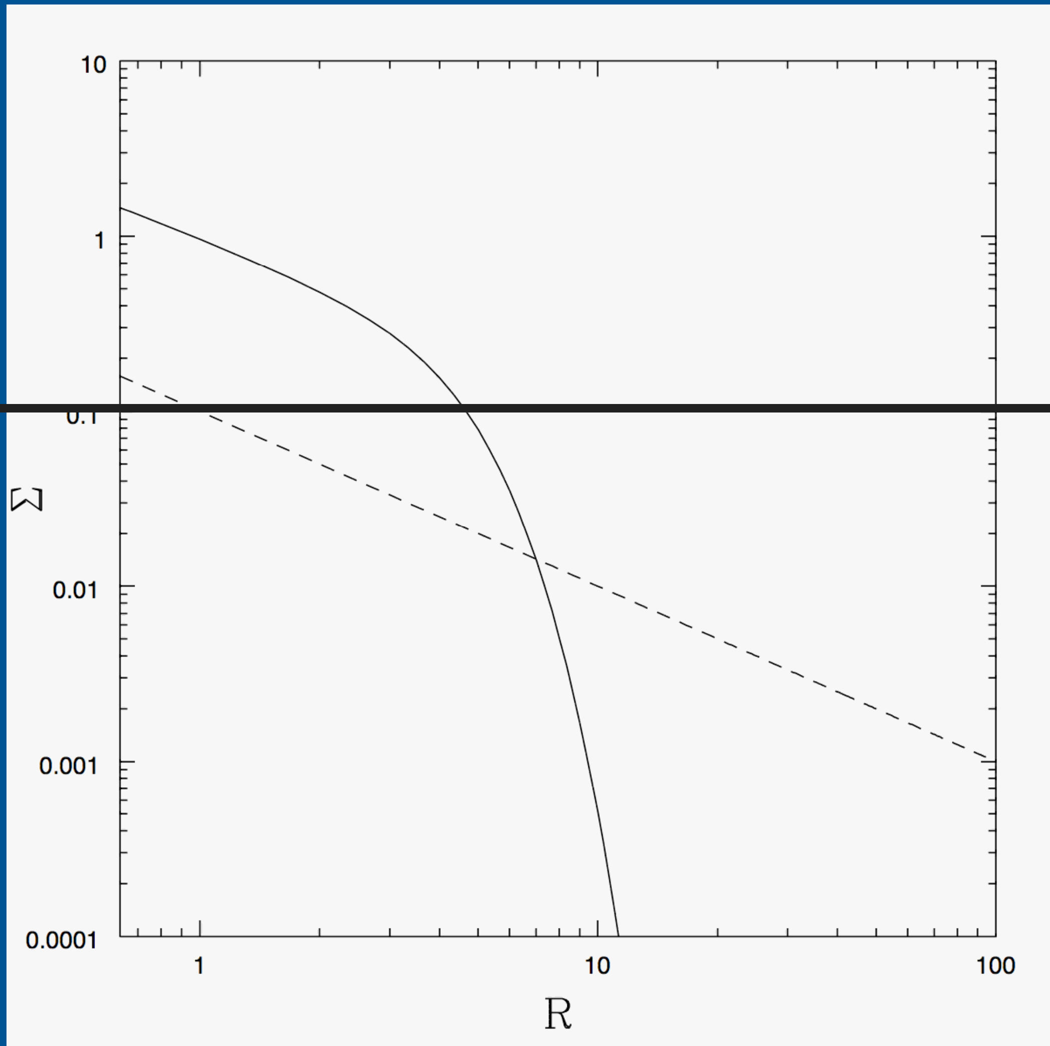
Self-similar solutions

- Analytical solutions when ν is a power-law (Lynden-Bell & Pringle 1974, Hartmann 1998)
- Here, $\nu \sim R$
- At small radii, solution approaches steady state, with $\Sigma \sim R^{-1}$
- At large radii, exponential cut-off

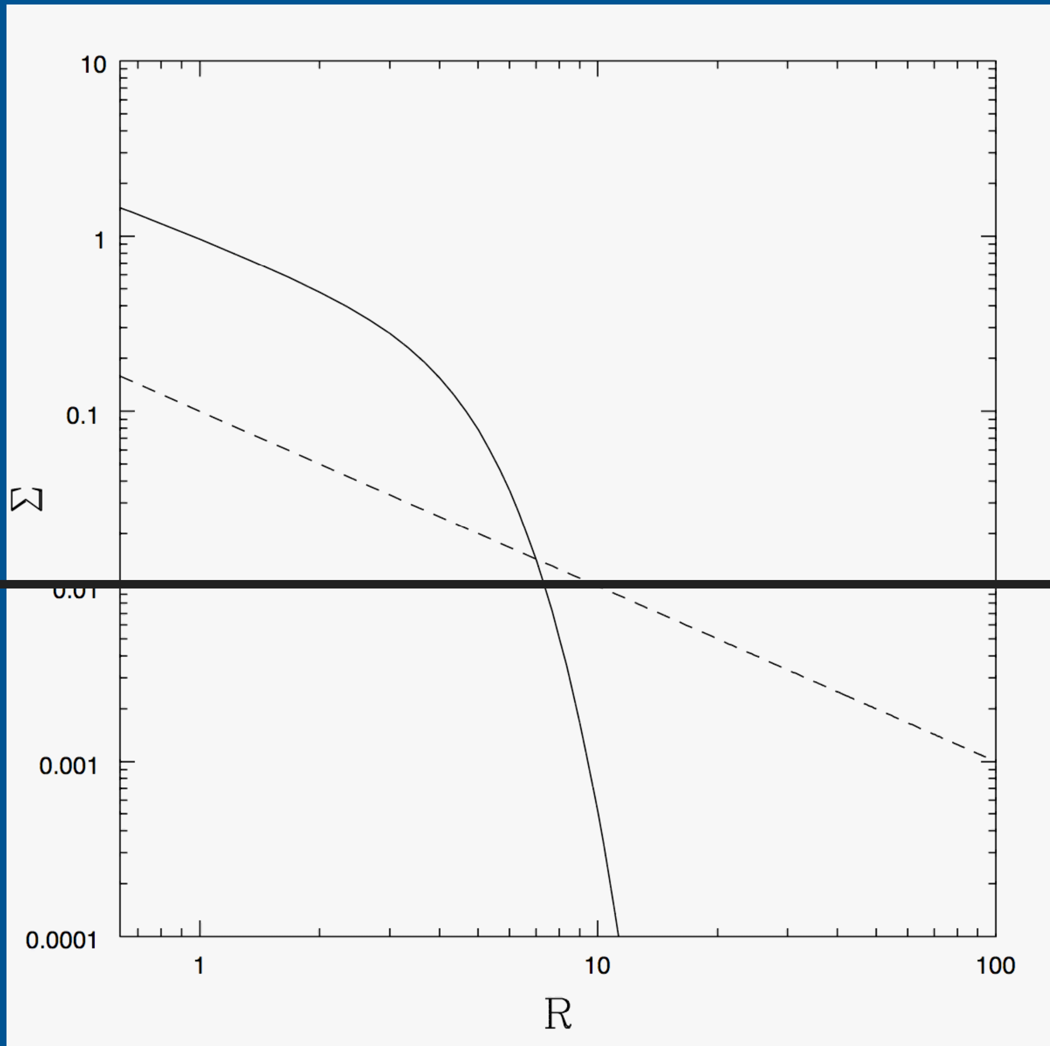
Self-similar solutions



Disc size vs age



Disc size vs age



Timescales

- Orbital motion (dynamical timescale)

$$t_{\rm dyn} = \Omega^{-1} = \sqrt{\frac{R^3}{GM}}$$

- Vertical hydrostatic balance $t_{\rm z} = H/c_{\rm s} = \Omega^{-1}$
- Thermal timescale

$$t_{\rm th} = \frac{\Sigma c_{\rm s}^2/(\gamma(\gamma-1))}{D(R)} = \frac{\Sigma c_{\rm s}^2/(\gamma(\gamma-1))}{\nu\Sigma(R\Omega')^2} = \frac{4}{9\gamma(\gamma-1)}\frac{1}{\alpha\Omega}$$

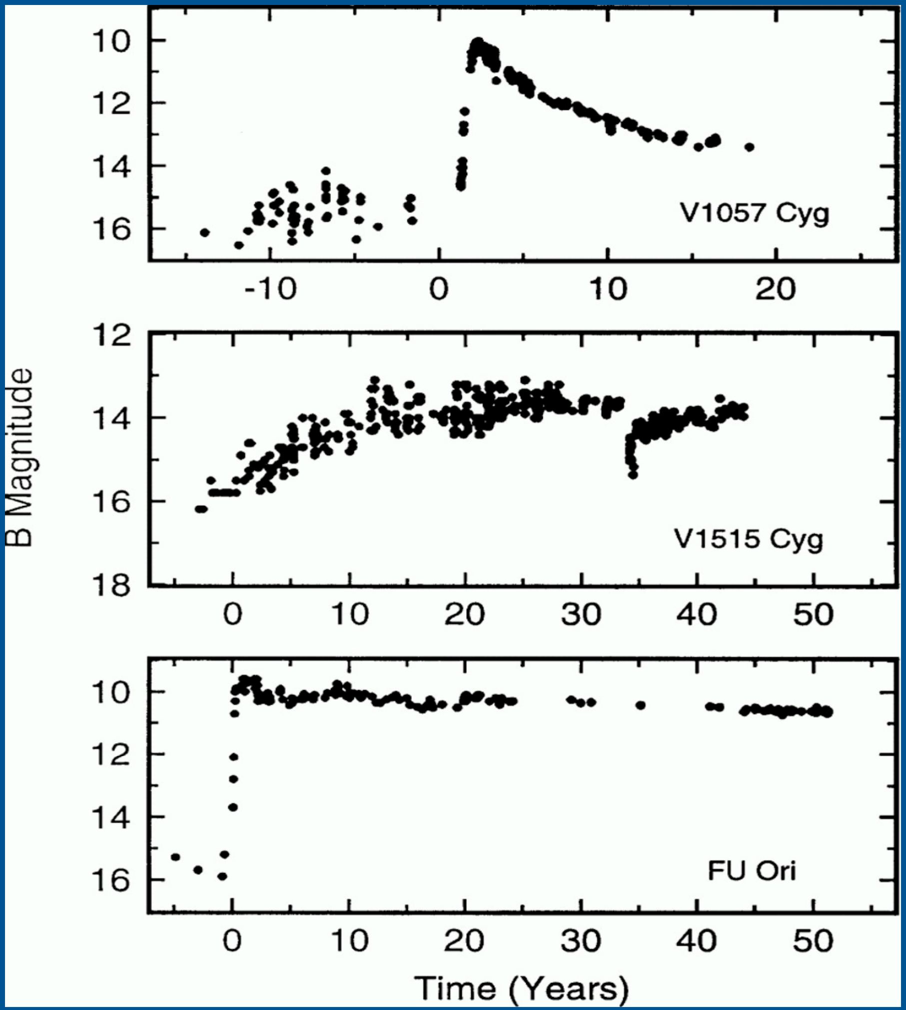
- Ordering

$$t_{\rm dyn} \sim t_z \ll t_{\rm th} = (1/\alpha)t_{\rm dyn} << t_{\rm visc} = (R/H)^2 t_{\rm th}$$

Thermal/viscous instability and FU Orionis

- FU Orionis: YSOs undergoing sudden outbursts
- In outburst: spectrum well described by active accretion disc (Kenyon & Hartmann)
- In quiescence: consistent with T Tauri

Thermal/viscous instability and

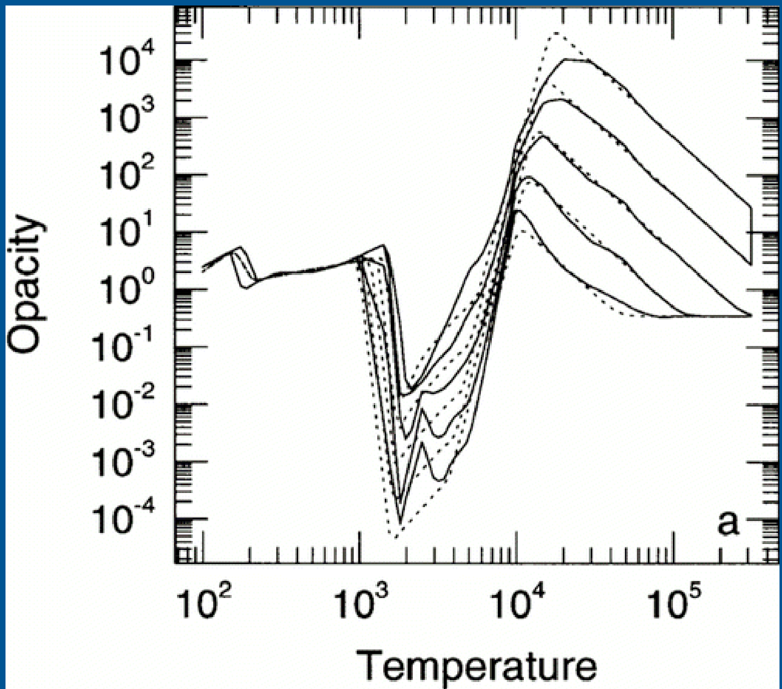


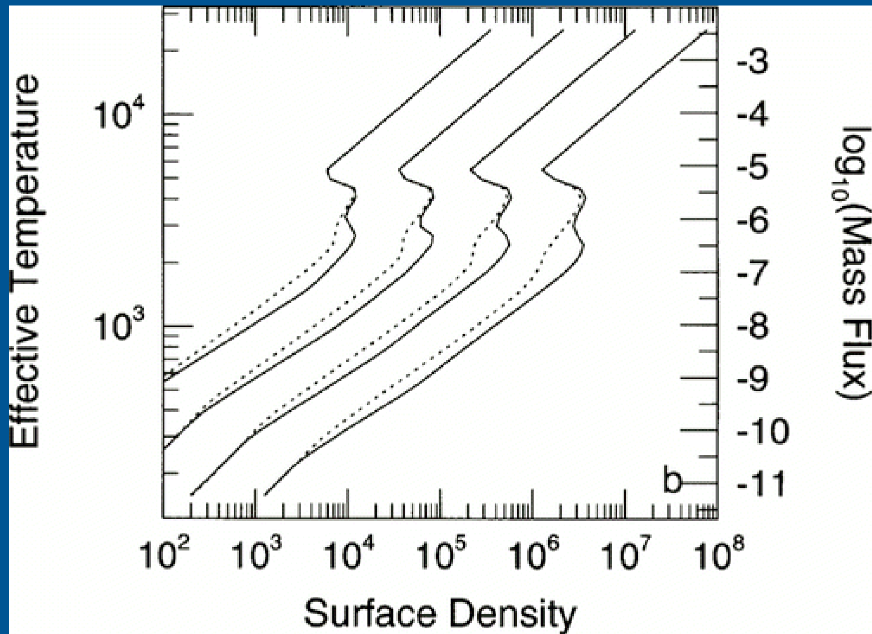
From Kenyon & Hartmann (1995)

Thermal/viscous instability and FU Orionis

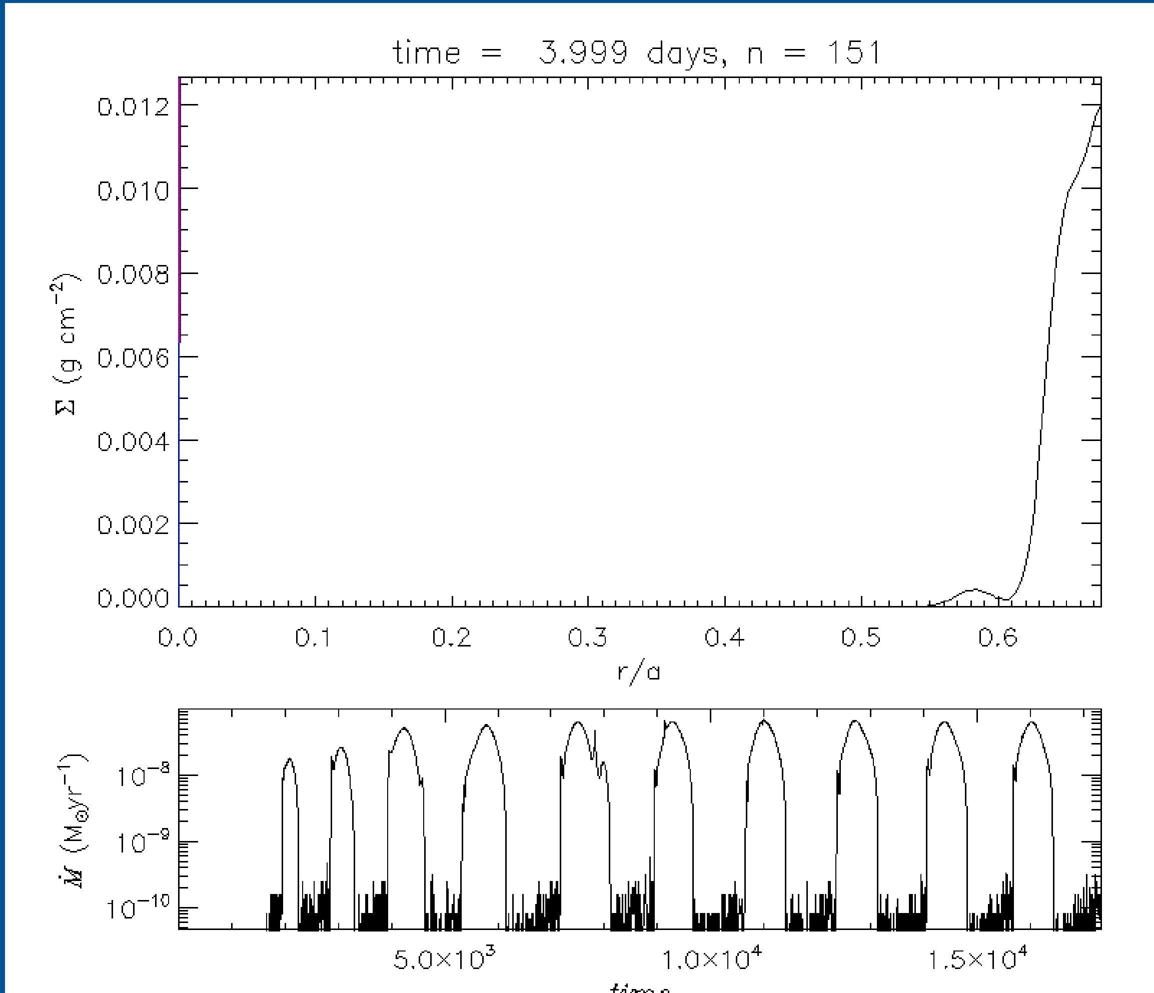
- At any radius: heating/cooling balance determines relation between mass flux and Σ
- Realistic opacities: gap at 10^3-10^4 K (Bell & Lin 1994)
- Negative slope in M_{dot}-Σ: instability!
- Disc can only stay either in
  - the upper branch: high accretion, outburst
  - or in the lower one: low accretion, quiescence

Thermal/viscous instability and FU Orionis





Limit cycle instability in CVs



Courtesy of Patrick Deegan and Graham Wynn

Limit cycle in FU Ori?

- Many models proposed:
  - Clarke et al. (1990), Lodato & Clarke (2004)
  - Bell & Lin (1994), Bell et al. (1995)
- Problems:
  - Hard to match different timescales
  - Triggered/untriggered outbursts
  - Required values of $\alpha \sim 10^{-3}$-10^{-4} very small